Solvable model in renormalization group analysis for effective eddy viscosity

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This study presents a solvable model in renormalization group analysis for the effective eddy viscosity. It is found fruitful to take a simple hypothesis that large-scale eddies are statistically independent of those of smaller scales. A limiting operation of renormalization group analysis yields an inhomogeneous ordinary differential equation for the invariant effective eddy viscosity. The closed-form solution of the equation facilitates derivations of an expression of the Kolmogorov constant C_K and of the Smagorinsky model for large-eddy simulation of turbulent flow. The Smagorinsky constant C_S is proportional to $C_K^{3/4}$. In particular, we shall illustrate that the value of C_K ranges from 1.35 to 2.06, which is in close agreement with the generally accepted experimental values $(1.2\sim2.2)$.

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The development of renormalization-group (RG) analysis for turbulence has been difficult as many of the RG studies in one form of approach or another do not sound either mathematically or physically consistent. As Frisch [1] said in his notable book, ‘‘Twenty years later turbulence remains unsolved. However, RG methods stand a good chance of playing a role in the solution of problem of turbulence.’’

Nelkin [2] could possibly be among the earliest pioneers who studied the renormalization group theory of turbulence. But there are now basically two different RG approaches to fluid turbulence. One, known as ϵ-RG, was originated by Forster, Nelson, and Stephen [3] and has been developed by others [4]; and the other, known as recursive RG, was based on the works of Rose [5], Rose and Sulem [6] and also has been developed by some other authors [7], in particular, the works McComb and Watt [8,9] are most relevant to the present study. Among several outstanding difficulties that there is no closed-form solution for the effective eddy viscosity due to complicated recursive equations has prevented further development of recursive renormalization group analysis [10].

In this analysis, we start with the approach of recursive RG, examine the detailed procedure, and show where our proposed procedure deviates from the existing one. The key assumption made in the study is a typical hypothesis that large-scale eddies are considered to be statistically independent of those of smaller scales (cf. McComb [14], p. 356). This assumption is *simple but not void* though its validity may be restrictive. In our opinion, RG theory based on this assumption has not been explored to its full strength, and indeed significant new results can be obtained, as described in the abstract and in this text. In the meantime, reports on some successful applications of the present formulation of RG analysis to turbulent thermal transport [15] and magnetohydrodynamic turbulence [16] are available. In the present study, the flow turbulence considered is assumed to be isotropic, stationary, and homogeneous. The theory starts with the incompressible Navier-Stokes equation, which we have in the wave number space as (cf. McComb [14], p. 56)

$$\left(\frac{\partial}{\partial t}+\nu_0 k^2\right)u_\alpha(\mathbf{k},t)=\int d^3 j M_{\alpha\beta\gamma}(\mathbf{k})u_\beta(\mathbf{j},t)u_\gamma(\mathbf{k}-\mathbf{j},t). \tag{1}$$

The basic idea of recursive RG analysis is to divide the wave number space $(0,k_0)$, where k_0 is Kolmogorov's scale, in a supergrid $(0,k_c)$ and a subgrid region (k_c,k_0); the subgrid modes are then removed shell by shell by taking the subgrid average over a spherical shell (k_{n+1},k_n), as shown in Fig. 1.

Let us now proceed with the first step toward subgrid modeling, i.e., to consider the effect of removing the first subgrid shell $k_1<k<k_0$ from the Navier-Stokes equation in the RG procedure. To distinguish modes, we have to introduce the following notation:

$$u_\alpha(\mathbf{k},t)=\begin{cases} u_\alpha^<(\mathbf{k},t) & \text{for} \quad |\mathbf{k}|<k_1 \\ u_\alpha^>(\mathbf{k},t) & \text{for} \quad |\mathbf{k}|>k_1, \end{cases} \tag{2}$$

where the supergrid $u_\alpha^<(\mathbf{k},t)$ has support in $|\mathbf{k}|<k_1$, while the subgrid $u_\alpha^>(\mathbf{k},t)$ has support in $|\mathbf{k}|>k_1$. Now we adopt the typical assumption by taking the ensemble average over the particular subgrid shell modes under the consideration to obtain (cf. McComb [14], p. 356)

$$\langle u_\alpha^>(\mathbf{k},t)\rangle=0;\ \ \langle u_\alpha^<(\mathbf{k},t)\rangle=u_\alpha^<(\mathbf{k},t), \tag{3}$$

where the origin of the first one is the consideration of turbulence field with ensemble-mean-zero fluctuation, while the second one holds because the supergrid components are considered to be statistically independent of the subgrid average.

For the subgrid range $k_1<|\mathbf{j}|<k_0$, Eq. (1) becomes

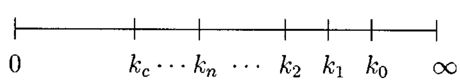

FIG. 1. The termini k_i for recursive renormalization with a fixed cutoff ratio $\Lambda=k_{n+1}/k_n$. Recursive renormalization analysis starts at the Kolmogorov's scale k_0, and ends at the cutoff wave number k_c.

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$$(\nu_0 j^2) u^>_\beta(\mathbf{j},t) = M_{\beta\beta'\gamma'}(\mathbf{j}) \int d^3j' [u^<_{\beta'}(\mathbf{j}',t) u^<_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + 2u^<_{\beta'}(\mathbf{j}',t) u^>_{\gamma'}(\mathbf{j}-\mathbf{j}',t) + u^>_{\beta'}(\mathbf{j}',t) u^>_{\gamma'}(\mathbf{j}-\mathbf{j}',t)]; \quad (4)$$

while for those k in the supergrid range of $|\mathbf{k}|<k_1$, Eq. (1) becomes

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u^<_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j [u^<_\beta(\mathbf{j},t) u^<_\gamma(\mathbf{k}-\mathbf{j},t) + 2u^<_\beta(\mathbf{j},t) u^>_\gamma(\mathbf{k}-\mathbf{j},t) + u^>_\beta(\mathbf{j},t) u^>_\gamma(\mathbf{k}-\mathbf{j},t)], \quad (5)$$

where we follow the Markovian approximation that in every rescaling, the subgrid velocity field $u^>_\alpha$ evolves faster than the supergrid velocity field $u^<_\alpha$ so that $\partial u^>_\alpha/\partial t$ can be neglected in Eq. (4). Let us observe that the second term on the right-hand side (RHS) of Eq. (5) has simply ensemble mean zero for the subgrid shell average under the hypothesis of statistical independence of the supergrid and the subgrid modes. It is therefore more natural to work directly with Eq. (5) by taking the subgrid average to obtain, without neglecting any contribution, the renormalized Navier-Stokes equation:

$$\left(\frac{\partial}{\partial t} + \nu_0 k^2\right) u^<_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j [u^<_\beta(\mathbf{j},t) u^<_\gamma(\mathbf{k}-\mathbf{j},t) + \langle u^>_\beta(\mathbf{j},t) u^>_\gamma(\mathbf{k}-\mathbf{j},t)\rangle]. \quad (6)$$

Equation (6) bears a close analogy with the original Navier-Stokes equation (1) except the second quadratic term on the RHS. It is our proposal that we work directly with Eq. (6) by looking into further details of the second term on the RHS of Eq. (6) to produce a workable form of the effective eddy viscosity.

Multiply both sides of Eq. (4) by $u^>_\gamma(\mathbf{k}-\mathbf{j},t)$ and then take the ensemble average over the subgrid shell modes. On the other hand, we make use of Eq. (4) by renaming the index β by γ and $\mathbf{j}$ by $\mathbf{k}-\mathbf{j}$, followed by multiplying on both sides by $u^>_\beta(\mathbf{j},t)$. Applying these results with proper rearrangement of the indices and change of variables to obtain

$$M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \langle u^>_\beta(\mathbf{j},t) u^>_\gamma(\mathbf{k}-\mathbf{j},t)\rangle = 4M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j \frac{M^>_{\gamma\beta'\gamma'}(\mathbf{k}-\mathbf{j})}{\nu_0|\mathbf{j}|^2+\nu_0|\mathbf{k}-\mathbf{j}|^2} D_{\gamma'\beta}(\mathbf{j}) \times D_{\alpha\beta'}(\mathbf{k}) Q(\mathbf{j}) u^<_\alpha(\mathbf{k},t). \quad (7)$$

Substitute Eq. (7) in Eq. (6) and rewrite the result as

$$\left(\frac{\partial}{\partial t} + \nu_1(k) k^2\right) u_\alpha(\mathbf{k},t) = M_{\alpha\beta\gamma}(\mathbf{k}) \int d^3j\, u_\beta(\mathbf{j},t) u_\gamma(\mathbf{k}-\mathbf{j},t), \quad (8)$$

where $\nu_1(k)$ is called the effective eddy viscosity after the first subgrid modeling,

$$\nu_1(k) = \nu_0 + \delta\nu_0(k).$$

After removing the nth shell, we have the relationship

$$\nu_{n+1}(k) = \nu_n(k) + \delta\nu_n(k), \quad (9)$$

with

$$\delta\nu_n(k) = \frac{1}{2\pi}\int_{\Omega_n} d^3j \frac{E_n(j) L(\mathbf{k},\mathbf{k}-\mathbf{j})}{j^2k^2[\nu_n(j)|\mathbf{j}|^2 + \nu_n(|\mathbf{k}-\mathbf{j}|)|\mathbf{k}-\mathbf{j}|^2]}, \quad (10)$$

where $\Omega_n(\mathbf{k}) = \{\mathbf{j} | k_{n+1} < |\mathbf{j}|,\ |\mathbf{k}-\mathbf{j}| < k_n\}$, and see the work of McComb [14] (p. 234) for $L(\mathbf{k},\mathbf{k}-\mathbf{j})$.

Next, we shall determine the energy spectrum for the entire range of wave numbers, which is supposed to be a combination form of the scaling laws proposed, respectively, by Pao [12] and Leslie and Quarini [13]:

$$E_n(j) = A_p\left(\frac{j}{k_p}\right) C_K \epsilon_n^a j^y \exp\left(-\frac{3}{2} C_K^{-1/2} \nu_n(j) \epsilon_n^b j^z\right), \quad (11)$$

where $A_p(x) = x^{s+5/3}/(1+x^{s+5/3})$; k_p is the wave number at which the spectrum is the maximum, a good choice for the exponent s is 4 (see, e.g., Leslie and Quarini [13]); C_K is the Kolmogorov constant; ϵ denotes the energy dissipation rate; and a, b, y, z are the undetermined parameters. In order to extract the correct dimension of $\nu_n(k)$ from Eq. (10), let us introduce a dimensionless variable $\boldsymbol{\eta} = \mathbf{j}/k$, then Eq. (10), with the proposed energy spectrum (11), becomes

$$\delta\nu_n(k) = \frac{k^{y-1}}{2\pi} \int_{\bar{\Omega}_n} d^3\eta \frac{\exp\left(\frac{-3}{2} C_K^{-1/2} \nu_n(j) \epsilon_n^b \eta^z k^z\right)}{\nu_n(k\eta)\eta^2 + \nu_n(k\zeta)(1+\eta^2-2\eta\mu)} \times \frac{A_p C_K \epsilon_n^a \eta^{y-2}(1-2\eta\mu+\eta^3\mu)(1-\mu^2)}{1+\eta^2-2\eta\mu}, \quad (12)$$

where we used the shorthand $\zeta = \sqrt{1+\eta^2-2\eta\mu}$ and μ denotes the direction cosine between $\mathbf{k}$ and $\mathbf{j}$. Equation (12) should give a consistent dimension on both sides. The only dimensional factors appearing in the numerator of the integrand are k and ϵ; this suggests that there is a dimensionless effective eddy viscosity $\hat{\nu}_n(\eta)$ such that

$$\nu_n(j) = \nu_n(k\eta) = C_K^p \epsilon_n^q k^r \hat{\nu}_n(\eta) \quad \text{for all } n. \quad (13)$$

Substituting Eq. (13) in Eq. (12) yields the following expression:

$$\delta\nu_n(k)=\frac{C_K^{1-p}\epsilon_n^{a-q}k^{y-1-r}}{2\pi}\times\int_{\bar{\Omega}_n}d^3\eta\,\frac{\exp\left(\frac{-3}{2}C_K^{-1/2}\nu_n\epsilon_n^b\eta^z k^z\right)}{\hat{\nu}_n(\eta)\eta^2+\hat{\nu}_n(\zeta)(1+\eta^2-2\eta\mu)}\times\frac{A_p\eta^{y-2}(1-2\eta\mu+\eta^3\mu)(1-\mu^2)}{1+\eta^2-2\eta\mu}. \tag{14}$$

This implies that

$$p=\frac{1}{2},\quad q=\frac{a}{2},\quad r=\frac{y-1}{2},$$

and thus

$$\nu_n(k)=C_K^{1/2}\epsilon_n^{a/2}k^{(y-1)/2}\hat{\nu}_n(\eta). \tag{15}$$

On the other hand, it is natural to require that the dissipation equation holds, $\int_0^{k_{n+1}}2\nu_n(k)k^2E_n(k)dk=\epsilon_n$, Substituting Eq. (11) and Eq. (15) into the dissipation equation gives $a=\frac{2}{3}$ and $y=-\frac{5}{3}$. Moreover, the above results should make the whole exponential factor dimensionless; this implies $b=-1/3$, $z=4/3$. Substitute all the values of the exponents in Eq. (11); the renormalized energy spectrum reads

$$E_n(j)=A_pC_K\epsilon_n^{2/3}j^{-5/3}\exp\left(\frac{-3}{2}C_K^{-1/2}\epsilon_n^{-1/3}\nu_n(j)j^{4/3}\right). \tag{16}$$

Equation (16) shows that the renormalized energy spectrum remains consistent with Kolmogorov's $-5/3$ law. It is the purpose of this study to look for the invariant effective eddy viscosity by pursuing a differential version of recursive RG analysis with the limiting operation $\Lambda\to1$. First of all, we rescale the wave number by setting $\tilde{k}=k/k_{n+1}$ and substitute this in Eq. (10), with use of Eq. (9), to obtain

$$k_{n+1}^t\tilde{\nu}_{n+1}(\tilde{k})=k_n^t\tilde{\nu}_n(\tilde{k}\Lambda)+k_n^{-8/3-t}\delta\tilde{\nu}_n(\tilde{k}\Lambda). \tag{17}$$

For consistency of the dimension on both sides of Eq. (17), we must have $t=-4/3$. It is followed by division on both sides of Eq. (17) by $k_{n+1}^{-4/3}$,

$$\tilde{\nu}_{n+1}(\tilde{k})-\Lambda^{-4/3}\tilde{\nu}_n(\tilde{k}\Lambda)=\Lambda^{-4/3}\delta\tilde{\nu}_n(\tilde{k}\Lambda). \tag{18}$$

Now we write $\Lambda=1-\xi$ and let $n\to\infty$, equivalently, we have $\xi\to0$ and $\tilde{\nu}_n\to\tilde{\nu}$, then Eq. (18) becomes, for $n\gg1$,

$$\tilde{k}\frac{d\tilde{\nu}(\tilde{k})}{d\tilde{k}}+\frac{4}{3}\tilde{\nu}(\tilde{k})=C_K\epsilon^{2/3}A_p\left(\frac{1}{\tilde{k}_p}\right)\frac{\tilde{k}}{4}\left[1-\left(\frac{\tilde{k}}{2}\right)^2\right]\times\frac{\exp[-1.5C_K^{-1/2}\tilde{\nu}(1)\epsilon^{-1/3}]}{\tilde{\nu}(1)}, \tag{19}$$

which is an inhomogeneous ordinary differential equation, sought for the invariant effective eddy viscosity $\tilde{\nu}(\tilde{k})$. The equation is particularly useful because it can be exactly solved to yield

$$\nu(k)=\left[\nu(k_c)k_c^{4/3}-\frac{135}{364}M(k_c)\right]k^{-4/3}-M(k_c)\left[\frac{3}{52}\left(\frac{k}{k_c}\right)^3-\frac{3}{7}\left(\frac{k}{k_c}\right)\right]k_c^{-4/3}, \tag{20}$$

where

$$M(k_c)=C_K\epsilon^{2/3}A_p\left(\frac{k_c}{k_p}\right)\frac{\exp[-1.5C_K^{-1/2}\epsilon^{-1/3}\nu(k_c)k_c^{4/3}]}{4\nu(k_c)k_c^{4/3}}.$$

The purpose is to determine the value of ν at k_c. If $k_c/k_0\ll1$, we can make the approximation $A_p\approx1$ and obtain

$$\nu(k_c)\approx C_K^{1/2}\epsilon^{1/3}\left[F(k_0)k_0^{4/3}-\frac{135}{364}\frac{\exp[-1.5F(k_0)k_0^{4/3}]}{4F(k_0)k_0^{4/3}}\right]k_c^{-4/3}. \tag{21}$$

Let us write $\nu(k)=C_k^{1/2}\epsilon^{1/3}F(k)$, we have $\nu(k_c)=C_K^{1/2}\epsilon^{1/3}F(k_c)$, and therefore the above result is equivalent to the approximation

$$F(k_c)k_c^{4/3}\approx F(k_0)k_0^{4/3}-\frac{135}{364}\frac{\exp[-1.5F(k_0)k_0^{4/3}]}{4F(k_0)k_0^{4/3}}. \tag{22}$$

Let us define $B(\tilde{k})=C_K^{-1/2}\epsilon^{-1/3}k_c^{4/3}\nu(k)$ and rearrange it to $\nu(k)=B(\tilde{k})C_K^{1/2}\epsilon^{1/3}k_c^{-4/3}$. Replace k_c by k_0 in $\nu(k)$ and then evaluate ν at $k=k_0$ to obtain the molecular viscosity ν_0 in the form

$$B(1)C_K^{1/2}\epsilon^{1/3}k_0^{-4/3}=\nu_0=C_K^{1/2}\epsilon^{1/3}F(k_0)$$

where the second equality comes simply from the definition of $F(k)$. Comparison between both sides gives $F(k_0)k_0^{4/3}=B(1)$. The value of $B(1)$ is now estimated according to the Edwards-Fokker-Planck theory (cf. McComb [14], p. 268), which implies a constraint that our notation can be written as $C_KD/(B(1)C_K^{1/2})^2=1$, where D is a constant. According to this testing field model, Kraichnan [11] has the estimate $D=0.44$, whence we have $B(1)=\sqrt{0.44}=0.6633$, and therefore

$$F(k_c)k_c^{4/3}\approx B(1)-\frac{135}{364}\frac{e^{-1.5B(1)}}{4B(1)}=0.6116.$$

Finally, we obtain by substituting this result in Eq. (21),

$$\nu(k_c)\approx0.6116C_K^{1/2}\epsilon^{1/3}k_c^{-4/3}. \tag{23}$$

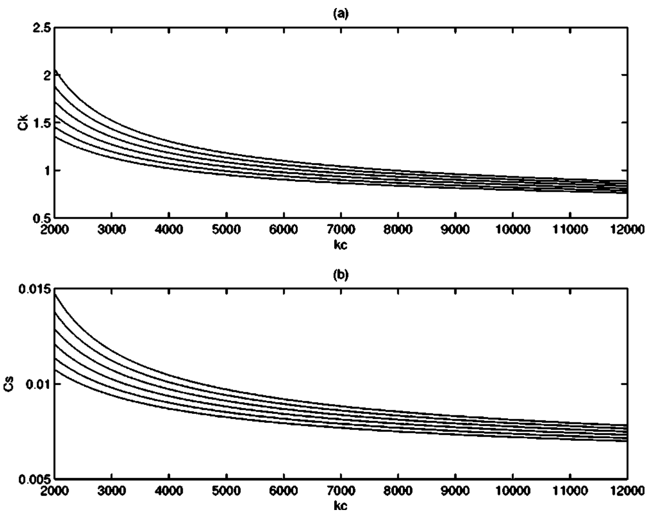

FIG. 2. A plot of the Kolmogorov and the Smagorinsky constants for $k_s = 450$ at various cutoff sizes. (a) The Kolmogorov constant C_K versus the cutoff wave number k_c; (b) the Smagorinsky constant C_S versus the cutoff wave number k_c. Each curve is further specified by the wave number k_p of the energy peak, which ranges from 500 to 1000; the upper curve corresponds to higher k_p.

Next by recalling Eq. (16), we can now obtain the Kolmogorov constant through substituting $\nu(k) = C_K^{1/2}\epsilon^{1/3}F(k)$ in the energy dissipation equation; this leads to

$$C_K = \left\{ 2\int_{k_s}^{k_c} F(k) A_p k^{1/3} \exp[-1.5F(k)k^{4/3}]dk \right\}^{-2/3}. \tag{24}$$

Numerical evaluation of the integral for various k_c, k_p, and k_s enables us to have a close look at the dependence of C_K. Figure 2(a) shows the plotted result. The behavior of C_K is observed to be dependent upon the value of k_p; the higher the value of k_p, the higher is the C_K curve. On the left end of the figure, the wave number is conceived to be falling in the inertial subrange; the value of C_K lies between 1.35 and 2.06, which is comparable to the generally accepted experimental value 1.2–2.2 (cf. McComb [14], p. 379). It is also of interest to integrate Eq. (24) using Eq. (20) by neglecting the second part on the RHS. This gives

$$C_K = \left\{ \frac{0.4887}{s+\frac{5}{3}} \left[\ln \frac{(k_c/k_p)^{s+5/3}+1}{(k_s/k_p)^{s+5/3}+1} \right] \right\}^{-2/3}. \tag{25}$$

The approach of recursive renormalization is naturally linked to the ideas of (effective) eddy viscosity and subgrid-scale modeling. The section is devoted to the derivation of the Smagorinsky model and determination of the Smagorinsky constant. Following Yakhot and Orszag [4] we first substitute C_K in $\nu(k_c)$ and express ϵ in the resolvable velocity,

$$\epsilon = \frac{\nu(k_c)}{2} \left(\frac{\partial \mathbf{u}_i^<}{\partial \mathbf{x}_j} + \frac{\partial \mathbf{u}_j^<}{\partial \mathbf{x}_i} \right)^2,$$

which yields with the use of Eq. (21), the Smagorinsky constant

$$C_S = \frac{1}{4\sqrt{2}\pi^2}(0.6633 C_K^{1/2})^{3/2} = 0.0097 C_K^{3/4}. \tag{26}$$

In parallel to Fig. 2(a) for C_K of Eq. (25), Fig. 2(b) shows the corresponding plot for the Smagorinsky constant C_S of Eq. (26). Since C_S is simply proportional to $C_K^{3/4}$, the Smagorinsky constant C_S is expected to behave similarly as the Kolmogorov constant C_K. That is, C_S should in general be the function of the cutoff size Δ as well as the wave numbers k_p and k_s, which correspond, respectively, to the energy-containing eddies and the largest-scale eddies existing in the flow.

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